Fourteen Proofs of a Result About Tiling a Rectangle

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STAN WAGON: I received my undergraduate degree at McGill and my doctorate at Dartmouth, in set theory under James Baumgartner. Recently my work has centered around expository writing: *The Banach-Tarski Paradox* was published in 1985 (Cambridge), and a series of eight articles on numerical evidence for various conjectures appeared in *The Mathematical Intelligencer* in 1985–6.

1. Introduction. In [**2**] (see also [**5**]) N. G. de Bruijn proved a result about packing n-dimensional bricks into an n-dimensional box that, when $n = 2$, implies that if an $a \times b$ rectangle is tiled with copies of a $c \times d$ rectangle, then each of c, d divides one of a, b. By a *tiling* we mean a covering with interior pairwise-disjoint sets. De Bruijn's proof has been generalized to yield the following more general theorem (illustrated in Figure 1), which implies his result on bricks (in the case $n = 2$, divide each side of the box by c (resp., d)).

THEOREM 1. *Whenever a rectangle is tiled by rectangles each of which has at least one integer side, then the tiled rectangle has at least one integer side.*

At the 1985 Summer Meeting of the MAA in Laramie, Wyoming, Hugh Montgomery mentioned this theorem and the proof using double integrals, in the hope of stimulating a search for more elementary proofs. That he did, as proofs have been forthcoming from various countries. Indeed, the variety of techniques that have been brought to bear is striking. Paul Erdös [**1**, p. 87] has suggested that "[God] has a transfinite book of theorems in which the best proofs are written." It is by no means clear which of the many proofs that follow is the best (the criteria for inclusion in the book are not readily available!). Perhaps none of these proofs is in the book, and the "best" proof has yet to be discovered. Even if simplicity is taken as the criterion, it is not completely clear which proof wins—the checkerboard and bipartite graph proofs seem to be the top candidates. And if strength is taken into account, that is, the ability to yield, perhaps with modification, more general results, then the situation is complicated. Variations of the theorem are true on the cylinder and torus, in higher dimensions, and for multiple tilings, but no one of the proofs is best in terms of its ability to generalize. Before reading Section 3 the reader might enjoy trying to predict which of the proofs are most likely to generalize.

Max Zorn has pointed out that Dehn considered similar questions in 1903. Dehn [**3**, p. 327] proved, as a corollary to a rather different sort of investigation, that if a rectangle is tiled as in Theorem 1, then one of the sides is rational.

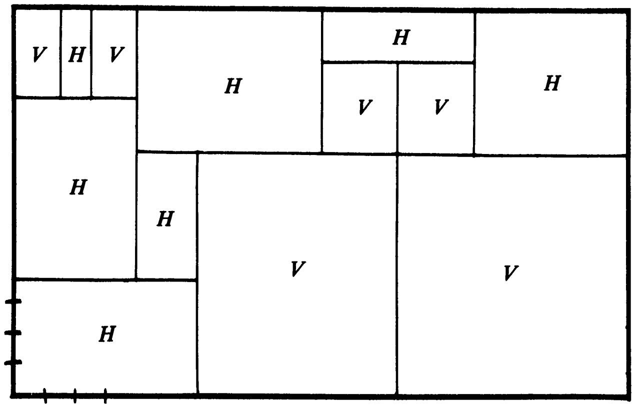


FIG. 1. An example of a tiling in which each tile has at least one integer side. The tiles labelled "H" have integer width; those labelled "V" have integer height.

2. The proofs. The *width* (resp., *height*) of a rectangle denotes its horizontal (resp., vertical) dimension. Given a tiling as in Theorem 1, let R denote the ambient rectangle. Let a tile with integer width be called an *H-tile* ("horizontal tile"); the other tiles, necessarily having integer height, are called *V-tiles* ("vertical tiles"). It is often assumed that R is in *standard position*, that is, its lower left corner is at the origin and its sides are parallel to the coordinate axes in the x-y plane.

(1) *Complex double integral* (extends original method of de Bruijn) First observe that $\int_a^b \sin 2\pi x\, dx = 0$ if and only if one of $a \pm b$ is an integer and $\int_a^b \cos 2\pi x\, dx = 0$ if and only if one of $a - b$, $a + b - 1/2$ is an integer. It follows that for any rectangle T in the x–y plane with sides parallel to the axes,

$$\iint_T e^{2\pi i(x+y)} dA = 0$$

if and only if at least one side of T has integer length. Now, the hypothesis implies that the double integral over each tile vanishes and therefore, by additivity of integrals, the double integral over R is zero. This implies that either the width or height of R is an integer. ■

(2) *Real double integral* (variation of complex double integral proof) Assume R is an $a \times b$ rectangle in standard position. As in the preceding proof, $\iint_T \sin 2\pi x \sin 2\pi y\, dA = 0$ for each tile T. Therefore, the double integral over R vanishes, which, because R has a corner at $(0,0)$, implies that at least one of a, b is an integer. (One could use other integrands as well, for example, $(x - [x] - 1/2)$, $(y - [y] - 1/2)$.) ■

(3) *Checkerboard* (Richard Rochberg, Washington Univ.; Sherman K. Stein) Place R in standard position. Color the square lattice generated by a $(1/2) \times (1/2)$ square with lower left corner at $(0,0)$ in black/white checkerboard fashion. Since

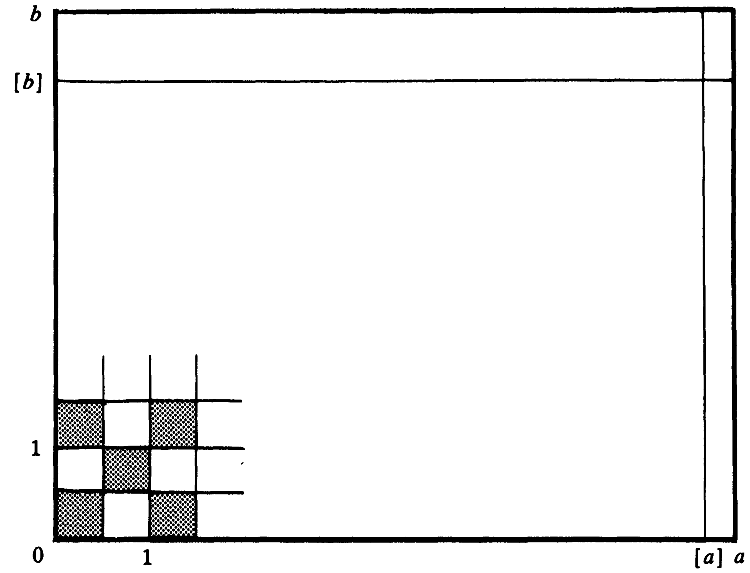


FIG. 2. If neither a nor b is an integer then the upper right corner has more black than white.

each tile has an integer side, each tile contains an equal amount of black and white. Therefore, the same is true of R. But then R must have an integer side for otherwise it can be split into four pieces (see Figure 2), three of which have equal amounts of black and white while the fourth does not. (This proof is derived from the preceding proof by using the integrand $(-1)^{[2x]}(-1)^{[2y]}$.) ■

(4) *Counting squares* (Imre Z. Ruzsa, Mathematical Institute of the Hungarian Academy of Sciences, Budapest; Peter Gilbert, Digital Equipment Corp., Nashua, NH) Place R in standard position and let $\{x_i\}$ (resp., $\{y_j\}$) be the set of x-coordinates of vertical (resp., y-coordinates of horizontal) boundary lines of tiles. Construct an auxiliary tiling (of a possibly new rectangle R') by translating all line segments in R's tiling as follows. If a segment is on a line corresponding to an integer value of x_i or y_j, it is not moved. If it is a vertical segment lying on $x = x_i$, where x_i is not an integer, translate it rightward or leftward to the line $x = [x_i] + 1/2$. Similarly, vertical segments on $y = y_j$ are translated up or down to $y = [y_j] + 1/2$, if y_j is not an integer. This construction may reduce the number of tiles, but this is unimportant.

Now, if the conclusion is false then R' is a rectangle in standard position having both side-lengths equal to one-half of an odd integer. Hence, R' contains an odd number of squares in the (uncolored) checkerboard described in the previous proof. But the hypothesis implies that each tile in R' has an even number of squares, contradiction. ■

(5) *Polynomials* (Adrien Douady, École Normale Supérieure, Paris) Place R in standard position and construct an auxiliary tiling in a way similar to the preceding proof. Choose a parameter t and translate only those segments having a noninteger coordinate. Translate vertical segments on $x = x_i$ rightward to $x = x_i + t$, and horizontal segments upward to $y = y_j + t$. If t comes from a sufficiently small interval $[0, \varepsilon]$, this construction yields a tiling of R', with the same number of tiles as in R.

Now, if the conclusion is false then R' is an $(a + t) \times (b + t)$ rectangle whence its area is a quadratic polynomial in t. But the hypothesis implies that each $w \times h$ tile in R becomes, in R', a tile of one of the forms $w \times (h \pm t)$, $(w \pm t) \times h$, $w \times h$. In all cases the area of the modified tile is a linear or constant function of t, and, hence, the same is true of the area of R'. Since t can take on any value in an interval, this contradicts the quadratic representation of the area. ■

(6) *Prime numbers* (Raphael Robinson, Univ. of California, Berkeley) We claim that for each prime p, either the height or width of R is within $1/p$ of an integer. It follows that one of these is an integer. To prove the claim, scale the entire tiling up by a factor of p in each direction, and consider the tiling obtained by replacing all tile-corners (x, y) in the scaled-up tiling by $([x],[y])$. This yields an integer-sided rectangle tiled by integer-sided rectangles, each of which has one side a multiple of p. Therefore, the area of the large integer-sided rectangle is a multiple of p, whence one of its sides must be a multiple of p. Moreover, the dimensions of this rectangle differ from the dimensions of the scaled-up rectangle by less than 1. It follows that R has a side that differs from an integer by less than $1/p$. ■

(7) *Eulerian path* (Michael S. Paterson, Univ. of Warwick, Coventry, England) Let Γ be the graph whose vertices are the corners of all the tiles, with two vertices joined whenever they correspond to the ends of a horizontal side of an H-tile or the vertical side of a V-tile. Multiple edges may exist. To make the picture clearer (and to see that Γ is planar), curve the edges a little in the direction of the tile defining the edge (see Figure 3). All vertices (except the corners of the large rectangle) lie on either 2 or 4 rectangles, and hence on either 2 or 4 edges in Γ. The corner vertices lie on 1 edge. It follows that a walk along edges that begins at one corner and does not repeat any edges will not terminate until it hits another corner, thus proving Theorem 1. ■

(8) *Bipartite graph* (variation of Eulerian path proof) Place R in standard position, let S be the set of corners of tiles having both coordinates integers, and let T be the set of tiles. Form a bipartite graph on $S \cup T$ by connecting each point in S to all tiles of which it is a corner. There is an even number of edges because the hypothesis implies that each tile has 0, 2, or 4 corners in S. But each point in S that is not a corner of R lies on either 2 or 4 tiles. Since $(0, 0)$, which lies on only one tile, is in S, there must be another point in S lying on an odd number of tiles. This can happen only if another corner of R lies in S, which means that either the width or height of R is an integer. ■

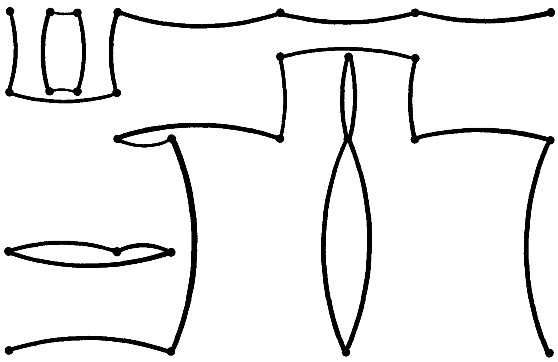


FIG. 3. The near-Eulerian graph Γ arising from the tiling in Figure 1.

(9) *Induction* (Raphael Robinson) The proof will be by induction on the number of H-tiles in a tiling in which each H-tile has width 1 and each V-tile has height 1. Since tiles may be split in their designated direction, this case suffices. Choose any H-tile T_0 (if there is none the result is immediate). If there are H-tiles whose lower border shares a segment with T_0's upper border, choose one and call it T_1. Otherwise only V-tiles share this border, and we may expand T_0 upward 1 unit. This does not increase the number of H-tiles, and the cut vertical tiles still have height 1. Continue expanding T_0 upward until either the top of the rectangle is reached, or a choice of an abutting H-tile T_1 is possible. Then continue upward similarly from T_1 to get T_2, etc. This yields a chain $T_0, T_1, \ldots, T_m$ of H-tiles from T_0 to the top of R. We can work downward from T_0 similarly, thus getting a chain

$$T_{-n}, \ldots, T_{-1}, T_0, T_1, \ldots, T_m$$

of H-tiles stretching from bottom to top. Remove these tiles and slide the rest together to get a rectangle with fewer H-tiles; induction applied to this smaller rectangle yields the result for the original rectangle. ■

(10) *Induction, variation* (Richard Bishop, Univ. of Illinois; Stan Wagon) Define a *V-link* to be a maximal vertical line segment in the tiling whose interior is not crossed by any horizontal line segment. Define *H-link* similarly. A link is *reducible* if it is a V-link (resp., H-link) having only H-tiles (resp., V-tiles) on one of its sides. In the tiling of Figure 1 there are lots of reducible links, for example, the V-link separating the large V-tile in the center from the two H-tiles on its left. It suffices to show that all tilings have a reducible link. For if we are given, say, a reducible V-link with only H-tiles bordering it on the right, let w be the width of the narrowest of these H-tiles. Then expand all tiles bordering the V-link on the left w units rightward (see Figure 4). Since heights are unchanged, V-tiles remain V-tiles; since widths are changed by the addition or subtraction of w, H-tiles remain H-tiles. But this expansion reduces the number of tiles by at least 1, as required for the induction.

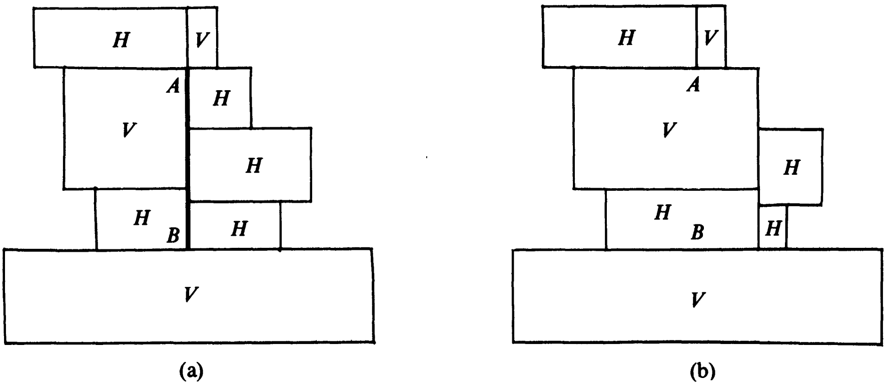


FIG. 4. In (a), AB is a reducible V-link. Figure (b) shows the tiling after the tiles adjacent to AB on the left have been expanded rightward.

A reducible link must exist, for otherwise there is a chain of H-tiles from bottom to top, each connected to the next along an H-link, and a chain of V-tiles from left to right, connected along V-links. The chains must cross, and the crossing must be an intersection of an H-link with a V-link in the interior of the links, contradicting the definition of a link. ■

(11) *Minimal cut-set* (Paul Seymour, Bell Communications Research, Morristown, NJ) Define a graph Γ as follows. The vertices are all horizontal line segments in the tiling, and two vertices are connected by m edges if there are m tiles (*either* H-tiles or V-tiles) connecting the corresponding segments. The exterior of R is considered as a tile, thus adding an additional edge connecting the top and bottom vertices. The tiling yields an embedding of Γ in the plane, since the vertical bisectors of the tiles can be used to form the edges (Figure 5). The edge corresponding to the additional tile can be drawn as in Figure 5, though it is more natural to preserve symmetry by embedding on the surface of a sphere instead.

Let Γ^* be the dual graph of Γ; the vertices of Γ^* are the faces of Γ and two vertices in Γ^* are connected by an edge if the corresponding faces in the planar embedding of Γ are incident. The faces of Γ have a simple structure: each face arises from part of a vertical segment in the tiling—a V-link, in the terminology of the preceding proof (see Figure 5)—and all tiles adjacent to the V-link. And if two faces in Γ are incident along an edge, then there is a tile whose vertical boundaries lie on the V-links corresponding to the faces.

Now let S be the set of edges in Γ corresponding to H-tiles, together with the exterior edge from top to bottom. If the removal of S does not disconnect the top vertex from the bottom vertex, then there is a top-to-bottom path with all vertical steps integers, as desired. Otherwise, let S' be a minimal subset of S whose removal disconnects the top from the bottom in Γ, and let S^* be the set of edges in Γ^* corresponding to edges in S'. By a well-known theorem for planar graphs [7, Thm.

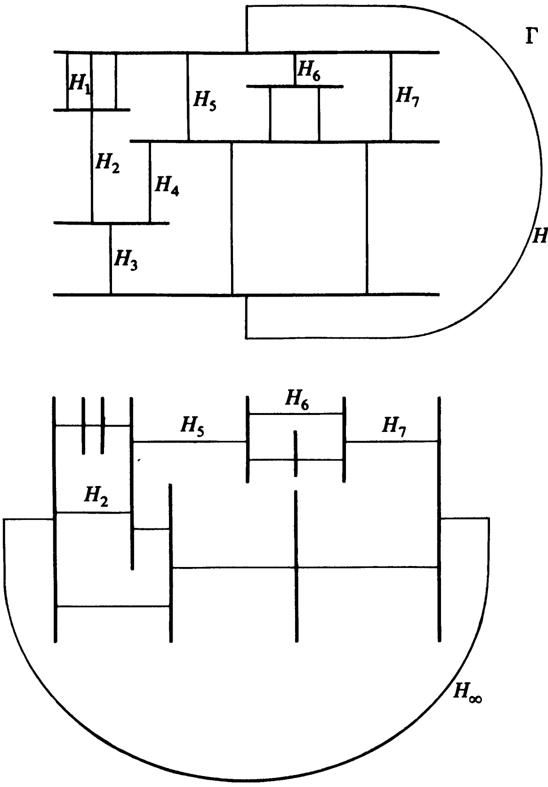


FIG. 5. The upper diagram represents the graph Γ of the minimal cut-set proof corresponding to the tiling in Figure 1. Horizontal segments are vertices, and vertical segments are edges, marked with an H if they arise from an H-tile in the tiling. The lower diagram shows the representation of the dual graph, Γ^*, using V-links in the tiling and horizontal segments as edges. The edges in S' and S^*, corresponding to a minimal cut-set in Γ, are H_2, H_5, H_6, H_7, and H_∞.

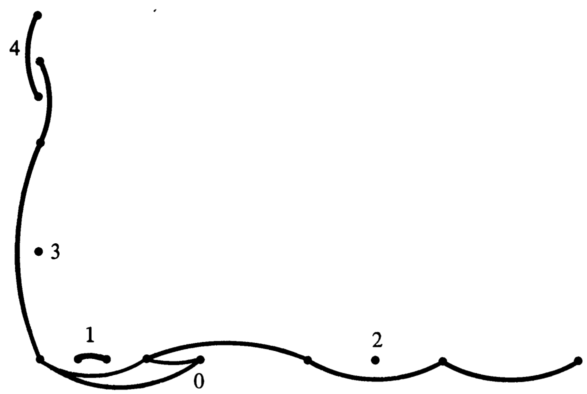
15C], S^* is a cycle in Γ^*. Moreover, since S' must contain the exterior edge, S^* induces a path from the left boundary of R to the right boundary, which has every horizontal step of integer length. Therefore, the width of the rectangle is an integer. ■

(12) *Sweep-line* (Gennady Bachman, Univ. of Illinois; Mihalis Yannakakis, Bell Labs, Murray Hill, NJ) Assume R is an $a \times b$ rectangle in standard position, and that b is not an integer. Let $\{R_i\}$ be the set of tiles, but assume that the closed segment forming the bottom border of each has been removed. Let a_i, b_i be the width and height, respectively, of R_i. Define f: $[0, b] \to [0, a]$ by setting $f(t)$ equal to the sum of all a_i such that R_i intersects the line $y = t$ and the y-coordinate of the top of R_i is not an integer. Then $f(0) = 0$ and it is easy to check that whenever f changes its value then it does so in a way that it remains an integer; as the

"sweep-line" crosses a horizontal line in the tiling the difference between f's gains and losses is an integer. Therefore, $f(b)$ is an integer. But since b is not an integer $f(b)$ is simply the sum of the widths of all tiles touching the top, that is, $f(b) = a$. ■

(13) *Step functions* (Melvin Hochster, Univ. of Michigan; Attila Máté, Brooklyn College) Place the rectangle in standard position. Then define a graph Γ whose vertices are all points on the x-axis such that some tile has a vertical boundary at that value (call these x_i, in increasing order), and all points on the y-axis that occur as top or bottom coordinates of some tile (y_i). Connect two vertices on the x-axis if some H-tile spans the interval and connect two vertices on the y-axis if some V-tile spans the interval (see Figure 6). The goal is to show that the origin lies in the same connected component of Γ as either $(a, 0)$ or $(0, b)$.

Assign, in an arbitrary way, distinct numbers to the connected components in Γ. Then define a step function on $[0, a]$ by defining f on the interval (x_i, x_{i+1}) to be



4	0	−4	0	0	8	−8	0
−4	0	4	0	0	−8	8	0
4	0	−4	0	0	8	−8	0
−3	0	3	0	0	−6	6	0
3	0	−3	0	0	6	−6	0

FIG. 6. The graph (with components numbered 0–4) and grid of the step function proof, using the tiling of Figure 1.

the number of the component that contains x_{i+1} less the number of the component that contains x_i. Note that the sum of the f-values on the intervals between two vertices connected by an edge is 0. Define g similarly on $[0, b]$. Now, refine the tiling into a grid by drawing all lines $x = x_i$ and $y = y_i$, and observe that $f(x)g(y)$ is constant in the interior of each rectangle in the grid. Moreover, the sum of these products over all grid rectangles contained in one of the tiles is 0 (see Figure 6). Therefore, the sum over all grid rectangles is 0. But this sum is just the product of $\Sigma\{f(I)$: I an interval between consecutive vertices on the x-axis$\}$ with $\Sigma\{f(J)$: J an interval between consecutive vertices on the y-axis$\}$. Therefore one of these sums vanishes, which implies that the origin and one of $(a,0),(0,b)$ lie in the same component. ■

(14) *Sperner's Lemma* (James Schmerl, Univ. of Connecticut) Assume the conclusion is false and R is placed in standard position. Triangulate R by drawing a diagonal in each tile. Then label all vertices in the tiling as follows: (x, y) is labelled A if $x \in \mathbf{N}$, B if $x \notin \mathbf{N}$ but $y \in \mathbf{N}$, and C if neither x nor y is an integer. Then by a variation to Sperner's Lemma (see [**6**, Lemma 2]), the number of triangles labelled ABC is odd. But the hypothesis implies that no triangle is so labelled, contradiction. ■

3. Generalizations. A first reaction to these proofs might be that they are not all different, since many of them have similar ingredients. In some cases this view is valid; the real double-integral proof is a specialization of the complex double-integral proof, and the checkerboard proof is a discretization of the real double-integral proof using a $\{\pm 1\}$-valued function instead of a product of sines. Also, the two induction proofs are closely related, as are the Eulerian path and bipartite graph proofs. But an examination of various generalizations brings out differences in all the other proofs (see Appendix).

A natural generalization of Theorem 1 is to the case where the integers are replaced by other groups of reals. Consider a tiling of R where each tile has one *designated* side, not necessarily of integer length (a tile with designated width (resp., height) is called an H-tile (resp., V-tile)). The goal here is to show that R has either its width in the (additive) subgroup of R generated by widths of H-tiles or its height in the group generated by heights of V-tiles. For example, if each tile has either integer width or algebraic length, then R has either integer width or algebraic length. The Eulerian path, minimal cut-set, sweep-line, step function, and polynomial proofs, as well as the variation to the induction proof, all yield this generalization with essentially no modifications. The bipartite graph proof works if S is the set of tile-corners having both coordinates in the corresponding groups. The induction proof can be made to work in this case, if one excises only part of the chosen horizontal tiles, corresponding to the width of the narrowest member, thus reducing the number of horizontal tiles.

Note that although the Eulerian path, minimal cut-set, and step function proofs all work by finding a path in a certain graph, there are essential differences. The first two use graphs that are planar, while the step function proof might construct a

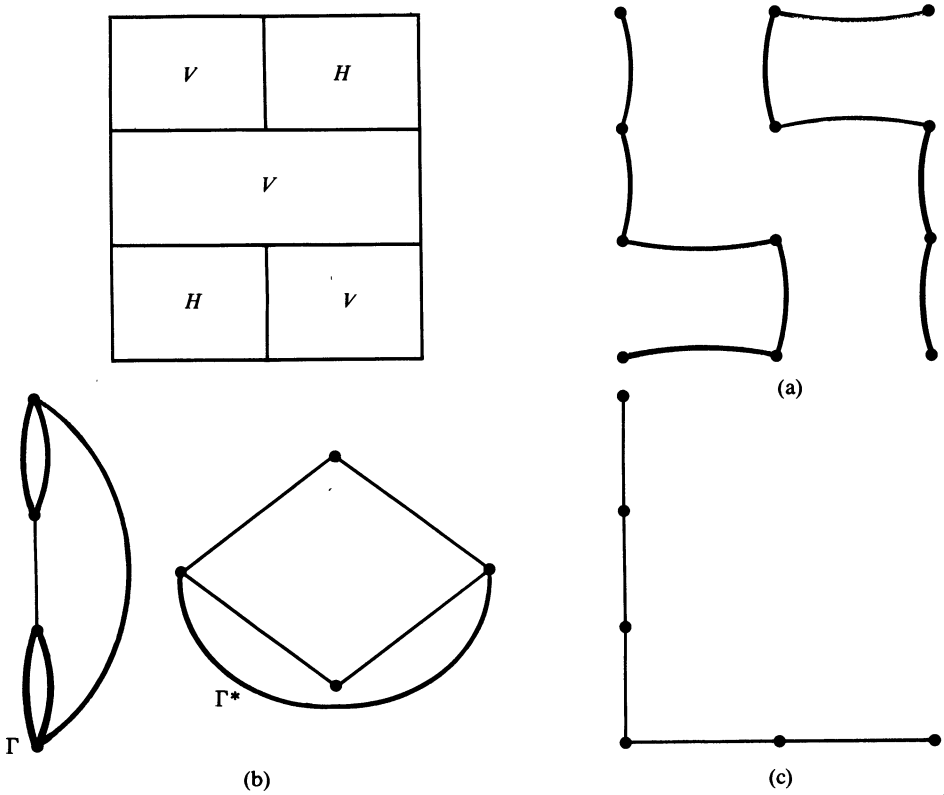


FIG. 7. Graph (a) is from the Eulerian path proof, the graphs labelled (b) are Γ and Γ* from the minimal cut-set proof, and graph (c) is from the step function proof.

nonplanar graph. The Eulerian proof is the only one of the three that shows that there is a path along integer-length sides of tiles from one side of the rectangle to the opposite side. However, the step function proof seems to have the capability of discovering "paths" that the others miss (see Figure 7).

Rusza has pointed out that Theorem 1 remains true only if it is assumed that tiles having at least one corner in $\mathbf{Z}^2$ have an integer side (here we assume R is in standard position). Another way of stating this is: Each tile has either 0, 2 or 4 corners in $\mathbf{Z}^2$. Rusza's square-counting proof, the bipartite graph proof, and the polynomial proof yield this result with no modification. The step function and Eulerian path proofs work as well (for the latter, consider only vertices lying in $\mathbf{Z}^2$), as does the Sperner Lemma proof.

As observed by several of the authors of proofs of Theorem 1, that result generalizes to higher dimensions. All the proofs, except (apparently) the minimal cut-set, sweep-line, and induction proofs, yield this generalization. Moreover, the higher-dimensional version allows k (rather than just 1) of the sides of each tile to be "designated."

THEOREM 2. *Suppose a box R in $\mathbf{R}^n$ is tiled with n-dimensional boxes and each tile has at least k integer sides. Then R has at least k integer sides.*

Proof. The polynomial proof requires almost no modification. The tiling can be perturbed by moving hyperplanes t units, for small t, as in the proof of Theorem 1. If the conclusion is false then the volume of the modified box is a polynomial in t having degree greater than k. But the hypothesis implies that each tile in the auxiliary tiling is a polynomial of degree at most k, contradiction. ∎

Several of the other proofs work as well. For the real integral proof, replace the integrand by the product of $t + \sin 2\pi x_r$, $r = 1, \ldots, n$. Then the integral over a box is a polynomial in t that is divisible by t^k if and only if the box has at least k integer sides. For the prime number proof, consider only primes p larger than any of the side-lengths. This guarantees that p^2 does not divide any of the side-lengths in the scaled-up box. The step function proof works if $f(x)$ and $g(y)$ are replaced by $t + f(x_1), t + f(x_2), \ldots,$ as in the extension of the integral proof, and a similar approach generalizes the checkerboard proof, which works easily if $k = 1$. The square counting proof works too, though if $k > 1$ one must use an odd integer when moving the boundary hyperplanes to an integer value; then the power of two dividing the number of squares in the auxiliary rectangle corresponds to the number of integer sides.

The Eulerian path and bipartite graph proofs yield Theorem 2 if $k = 1$, since each corner (except the corners of the ambient box) still lies on an even number of tiles. For larger k one can use induction, as pointed out by Andreu Mas-Colell: the $k = 1$ case yields one integer-length side; then project to the hyperplane perpendicular to this direction and use induction on the dimension. The advantage of this inductive approach is that it yields Ruzsa's extension for $k > 1$, where it is assumed only that tiles having a corner with all coordinates integers have k integer sides.

The polynomial, Eulerian path, and step function proofs of Theorem 2 show that the group-theoretical generalization to arbitrary n and k is valid. More precisely: If an n-dimensional box is tiled by boxes, each of which has at least k designated sides, then there are at least k directions in which the side-length of the ambient box lies in the subgroup of R generated by the designated side-lengths in the direction.

Another generalization comes from considering multiple tilings of the rectangle, that is, finitely many tiles that are not necessarily pairwise disjoint, but such that each point of the ambient rectangle (except for the boundaries of the tiles) is contained in the same finite number (the *multiplicity*) of tiles. The integration proofs work in the integer case, as do the checkerboard, polynomial, square counting (replace $1/2$ by $1/p$, where p is a prime larger than the multiplicity), and prime number proofs (use primes larger than the multiplicity). The Eulerian path proof will work if, as pointed out by Paterson, one makes a directed graph, with edges directed out of the lower left and upper right corners of each tile, and into the other

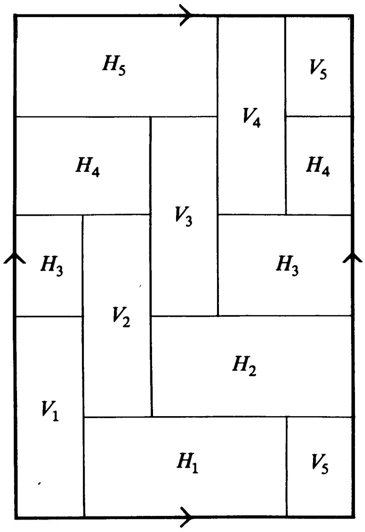


FIG. 8. H-tiles are 6×3; V-tiles are 2×6; torus is 10×15. Dividing by 6 yields a tiling of a $\frac{5}{3} \times \frac{5}{2}$ torus using $1 \times \frac{1}{2}$ and $\frac{1}{3} \times 1$ tiles.

two corners. Then the lower left corner has out-degree equal to the multiplicity, while the vertices not equal to a corner have equal in-degree and out-degree. Hence a directed walk starting from the lower left corner will end at one of the adjacent corners of R. The Eulerian path, step function, and polynomial proofs work in the case of groups as well.

Next, we can try to generalize to the case where sides of the rectangle are identified, that is, to the cylinder or torus. Consider the cylinder first, where we assume that opposite vertical sides are identified. The direct generalization of Theorem 1 is valid, as shown by either the sweep-line proof, the induction proof (which was invented for the cylinder and torus), the variation to the induction proof, or the Eulerian path proof (modified as in the proof of Theorem 3 below).

The torus is more interesting since Theorem 1 is false. Consider an $a \times b$ flat torus, that is, an $a \times b$ rectangle in the plane with opposite sides identified. The example in Figure 8, discovered independently by Solomon Golomb and Raphael Robinson, shows that the naive generalization of Theorem 1 is false.

Theorem 1 does nevertheless generalize to the torus, although the statement is more complicated. The following theorem was first proved in the integer case by Robinson, whose proof used the method of the induction proof of Theorem 1 and could be extended to the case of arbitrary subgroups. The proof of Theorem 3 given below combines ideas of the Eulerian path and induction proofs, and is due to Joan Hutchinson and the author. Note the curious situation that the original result in rectangles extends to arbitrary subgroups of $\mathbf{R}$, while the toroidal result generalizes to arbitrary subfields of $\mathbf{R}$.

THEOREM 3 (R. M. Robinson). *Suppose an a-by-b flat torus is tiled with rectangles parallel to the sides of the torus. Suppose each tile, regardless of its length or width, is designated to be either an H-tile or a V-tile and let G_H (resp., G_V) be the group generated by the widths of the H-tiles (resp., heights of the V-tiles). Then at least one of the following is true*:

(1) *a is in G_H*;

(2) *b is in G_V*;

(3) *For some relatively prime integers m and n, ma is in G_H and nb is in G_V.*

Proof. Let Γ be the graph associated with the tiling, as described in the Eulerian path proof. On the torus, loops can occur. To ensure that Γ embeds on the torus, curve the edges a little in the direction of the tile defining the edge; see Figure 9. As in the planar case, each vertex has degree 2 or 4. (In degenerate cases, such as a tiling with one tile, the corners have 2 or 4 loops.) Thus each component of Γ is Eulerian. In particular, any edge lies on a simple cycle.

The proof will be by induction on N, the total number of tiles. If $N = 1$ either (1) or (2) holds. For $N > 1$ observe that if Γ has a noncontractible cycle, then one of (1), (2) or (3) follows. For we may assume that there is a simple noncontractible cycle C. If C winds exactly once in one of the directions then (1) or (2) holds. Otherwise we may use the well-known result that if C winds more than once in one direction, then its winding numbers in the two directions are relatively prime (this is a consequence of P. Lévy's "Universal Chord Theorem" which implies that if $\gcd(m, n) = d$ then a simple curve from the origin to (m, n) has a chord that is a translate of the segment from the origin to $(m/d, n/d)$ (see [4, p. 23])). This yields (3). For example, the graph of the Robinson-Golomb tiling has two cycles, each winding thrice around the horizontal direction and twice vertically, so $3a$ is in G_H and $2b$ is in G_V. However, there are tilings for which Γ has no noncontractible cycles (example in Figure 10); in such cases we shall show that the tiling can be modified so that there are fewer than N tiles.

Suppose then that Γ has only contractible cycles. Then Γ must have a simple contractible cycle with no edges in its interior (called an *empty* cycle). For if C is a

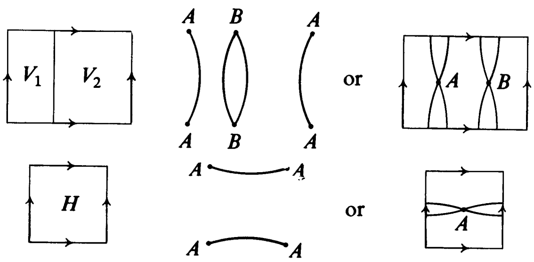


FIG. 9. Two examples—one with two tiles, one with just a single tile—of graphs associated with tilings of a torus.

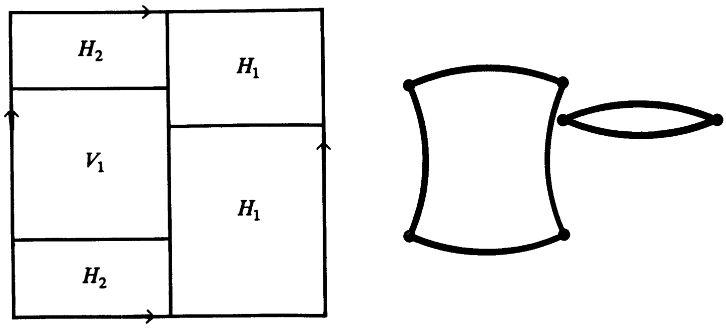


FIG. 10. An example of a toroidal tiling for which the graph has no noncontractible cycle.

simple contractible cycle with the fewest number of edges in its interior then this number must be 0; any edge inside C would lie on a cycle having fewer edges in its interior than C does.

First suppose Γ has an empty contractible cycle which, when viewed in the tiling, has no tile in its interior. Such a cycle, viewed in the tiling, must traverse each part of its boundary twice, once in each direction. Since the cycle is simple, this means it can have no right angles, and so must look like one of the cycles in Figure 11. In either case we can modify the tiles as in Figure 4 (expanding one side to absorb the narrowest—or shortest—tile on the other side), which reduces N by at least 1, as desired for the induction.

If there is no cycle as in the preceding paragraph, then Γ has an empty contractible cycle C that does have a tile, say, an H-tile, in its interior. Because C is empty, both the top and bottom of the tile correspond to edges on C. Label the tile's corners a, b, c, d starting from the upper right and going clockwise. Because C is a simple cycle, C must have the form $a \ldots bc \ldots da$. Now, adding the vertical steps in C between a and b yields that the distance from a to b lies in G_V. But then we may

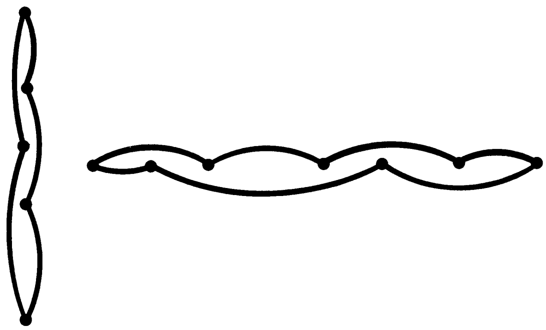


FIG. 11. A contractible cycle that, when viewed in the tiling, has no tile in its interior, must look like one of these two cycles.

switch the tile from an H-tile to a V-tile, shortening the length of the cycle. We can continue shortening the cycle in this way until it no longer has a tile in its interior. Then we are in the preceding case, where it was easy to reduce the number of tiles. ■

Not too much is known for higher-dimensional tori. Robinson has generalized his prime number proof to show that if such a torus is tiled with boxes having an integer side, then the torus has at least one rational side. This can also be proved by the step function proof, which has the advantage of working for arbitrary groups.

THEOREM 4 (Robinson, Mâté). *Suppose an n-dimensional flat torus with side-lengths a_i, $i = 1, \dots, n$, is tiled by n-dimensional boxes parallel to the sides of the torus, with each tile having one designated side. Let G_i be the group generated by the designated side-lengths in the ith direction. Then for at least one a_i there is a positive integer m such that ma_i is in G_i.*

Proof (Mâté). We give the details assuming the case of an $a \times b$ flat torus in $\mathbf{R}^2$; the extension to higher dimensions will be clear. Assume the $a \times b$ rectangle is in standard position and the origin is the corner of a tile. Extend the tiling periodically to the whole plane and define a graph Γ as in the step function proof: If (x, y) is a corner of a tile in any copy of the torus then the points $(x, 0)$ and $(0, y)$ are vertices. Connect two vertices on the x-axis if the interval they define is spanned by an H-tile in the tiling of the plane; this includes the case of tiles straddling a vertical boundary of a torus. Define edges on the y-axis likewise using V-tiles.

It is sufficient to prove the following claim, for if Γ has infinitely many vertices on, say, the x-axis that are in the same connected component, then that component must contain two vertices of the form $(x, 0), (x + ma, 0)$. This implies that ma is in G_1.

Claim. The graph Γ has an infinite connected component.

Proof of claim. To prove the claim, assume it is false. Then all components are finite and we may define a function C_1 on the points on the x-axis corresponding to vertices in Γ by letting $C_1(x)$ be the least t such that $(t, 0)$ is in Γ and in the same component as $(x, 0)$. Define C_2 for vertices on the y-axis similarly. Now define a step function f on the x-axis by letting f equal $C_1(x'') - C_1(x')$ on the interval between any two consecutive vertices x', x'' in Γ.

As before, the sum of f-values over intervals subdividing a single H-tile in the tiling of $\mathbf{R}^2$ is zero. Moreover, because of the periodicity of the tiling of the plane, $C_1(x + a) = C_1(x) + a$ and f is periodic with period a. It follows that the sum of f-values over intervals subdividing a single H-tile in the original torus in standard position is 0. These properties also hold for the step function g, defined using C_2 analogously to f. To conclude, proceed as in Theorem 1 to refine the tiling into a grid and observe that the sum of the fg values over a tile vanishes, whence the sum over the entire $a \times b$ torus vanishes. But this is a contradiction since this sum equals ab: the sum of f (resp., g) over the intervals in $[0, a]$ (resp., $[0, b]$) is simply $C_1(a) - C_1(0) = a$ (resp., $C_2(b) - C_2(0) = b$). ■

The preceding argument also works on a box in $\mathbf{R}^n$ where some, but not all, sides are identified. The result then states that either there is an "unidentified" direction of the box whose side-length is in the subgroup generated by designated lengths in that direction, or there is an "identified" direction for which an integer multiple of the side-length is in the group corresponding to that direction. In the case of the standard torus or cylinder this is not best possible, but the proofs in those cases do not generalize to higher dimensions. Unlike the proof of Theorem 3, the preceding argument works for multiple tilings, and so yields something for multiple tilings of the standard two-dimensional torus: If each tile has one integer side then at least one side of the torus is rational (and similarly in higher-dimensional multiple tilings).

4. Summary and open questions. The various generalizations considered here do a fairly complete job of distinguishing the proofs. If one calls two proofs equivalent provided they work on the same set of generalizations then, unless new modifications are found, the only equivalences are (1) ~ (2) ~ (3) and (9) ~ (10). The two most powerful proofs seem to be the Eulerian path and step function proofs. The former fails only on high-dimensional tori, multiple tilings of the standard torus, and the $k > 1$ case of Theorem 2; the latter works in all cases, except the cylinder and torus, where it does not yield the best possible result. A definitive comparison will have to wait until the true situation in higher dimensions is resolved; see Problem (a) below.

Problem (a). Can the seemingly weak statement about tilings of higher-dimensional tori or cylinders be improved, or is it best possible? The simplest unsolved case is that of a box with left and right faces identified. Can such a box having dimensions $a \times \beta \times \gamma$, where a is rational and β and γ are irrational, be tiled with boxes each of which has one side of unit length?

Problem (b) (S. Golomb). For which triples (a, b, k) can the $a \times b$ torus be tiled using copies (vertical or horizontal) of a $1 \times k$ tile?

REMARKS. De Bruijn's original result characterized the rectangles that could be tiled using copies of a $1 \times k$ tile: either the height or the width of the rectangle is a multiple of k. This is true for an $a \times b$ torus if k is a prime power. This was first proved by Robinson and Golomb using coloring techniques; it also follows from Theorem 3 above if one divides everything by k and observes that the relatively prime coefficients m and n cannot both absorb a power of the same prime. It follows that $k = 6$ is the smallest number for which there is a triple (a, b, k) as in Question (a) with neither a nor b divisible by k. An unsolved special case of Question (a) is the problem of determining which $a \times b$ tori can be tiled with copies of a 1×6 tile. Golomb has shown that the 10×15 torus is the smallest example.

Problem (c). What is the situation regarding double tilings of the standard torus where each tile has at least one integer side? Is it true that either one side of the torus is an integer or both sides are rational?

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Appendix to justify claim that proofs are different:

Proofs:

(1) Complex double integral
(2) Real double integral
(3) Checkerboard
(4) Counting squares
(5) Polynomials
(6) Prime numbers
(7) Eulerian path
(8) Bipartite graph
(9) Induction
(10) Induction, variation
(11) Minimal cut-set
(12) Sweep-line
(13) Step functions
(14) Sperner's Lemma

Generalizations:

1. Plane
2. Plane, Ruzsa hypothesis
3. Plane, arbitrary groups
4. n-dimensions, $k = 1$
5. n-dimensions, $k > 1$
6. n-dimensions, $k > 1$, Ruzsa hypothesis
7. Cylinder
8. Torus
9. Plane, multiple tiling
10. Plane, multiple tiling, arbitrary groups
11. High dimensional torus
12. Torus, multiple tiling

Proof number	Works in cases
1, 2, 3	1, 4, 5, 9
4	1, 2, 4, 5, 9
5	1, 2, 3, 4, 5, 9, 10
6	1, 4, 5, 9, 11
7	1, 2, 3, 4, 7, 8, 9, 10
8	1, 2, 3, 4, 9, 10
9, 10	1, 3, 7, 8
11	1, 3
12	1, 3, 7
13	1, 2, 3, 4, 5, 9, 10, 11, 12
14	1, 2, 3, 4