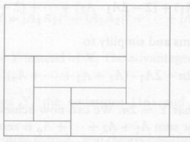
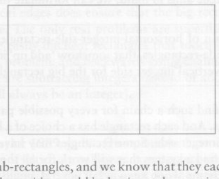
Problem 5.2. (*) A rectangle is partitioned into several smaller rectangles. Each of the smaller rectangles has at least one side of integer length. Prove that the big rectangle has at least one side of integer length.



This is a pleasant-looking problem, so presumably there should be a pleasant solution. But the conclusion is a bit odd: if all the small rectangles

have one integer side (or maybe more), why should the big rectangle have one? If we were not playing with rectangles, but just line segments, the thing would be easy: the big segment is composed of little segments, each of integer length, so the length of the big segment is a sum of integers which is an integer, of course. This one-dimensional case does not immediately offer any help to the two-dimensional case clearly, except we get the following clue: we have to use the fact that the sum of integers is an integer. One way we can use this fact immediately is to get some convenient notation: an 'integer side' means a side of integer length.

But the question also has traces of topology, combinatorics, and worse, mainly because of the word 'partition'. It is a bit too general. To get a handle on this question, let us try the simplest (but non-trivial) partitioning as follows:



We have two sub-rectangles, and we know that they each have at least one integer side. But those sides could be horizontal or vertical: we do not know which. Suppose the sub-rectangle on the left had a vertical integer side. But its vertical side length is the same as the big rectangle's vertical side length, so we have proved de facto that the big rectangle has an integer side. So we can assume that the left-hand rectangle has a horizontal integer side instead.

But we can argue by similar reasoning that the right-hand rectangle has a horizontal integer side, and so the big rectangle has a horizontal integer side, because it is the sum of two rectangles with horizontal integer sides. So we have proved this question for the special case of the two-rectangle partition. But how did it work? (Examples are really only useful when they give some insight into how the general problem works.) Scanning the above proof, we observe two key ingredients:

(a) We have to split into cases, because each sub-rectangle can have a vertical integer side or a horizontal integer side.

(b) The only way we can prove the big rectangle to have a vertical integer side, say, is if a 'chain' of smaller rectangles have a vertical integer side, and if the smaller rectangles somehow 'add up' to the big rectangle. Here

is an example, where the shaded rectangles have a horizontal integer side, so the big rectangle has to have a horizontal integer side as well:

So with these vague aims in mind, we can formulate this vague strategy:

Find either a chain of horizontal-integer-side-rectangles or a chain of vertical-integer-side-rectangles, that somehow 'add up' into a horizontal integer side or a vertical integer side for the big rectangle.

But we have to find such a chain for every possible partition. Partitions are very ugly things. And each rectangle has a choice of a horizontal integer side, or a vertical integer side. Some rectangles may have both. Now how could we possibly find a system that will work for all these possibilities?

How do these chains work anyway? If several small rectangles have a horizontal integer side, the big rectangle will have a horizontal integer side if the small rectangles 'link' from one end of the rectangle to another, as in the above figure, because then the length of the big edge is just the sum of the lengths of the smaller edges. (In other words, if you stack some blocks on top of each other, the total height of the structure is the sum of the height of the blocks.)

Part of the problem in finding these chains is that we do not know which rectangles have horizontal integer sides, and which rectangles have vertical integer sides. To visualize the possibilities, imagine that any rectangle with a horizontal integer edge is coloured green, and any rectangle with a vertical integer edge can be coloured red. (Rectangles with both horizontal and vertical integer edges are a bonus: they can be assigned either colour.) Now each small rectangle is coloured either green or red. And now we have to find a green chain connecting the two vertical edges or a red chain connecting the two horizontal edges.

No direct proof seems to be available, so let us try proof by contradiction. Suppose that the two vertical edges are not connected by green rectangles. Why cannot they be connected? Because there are not enough green rectangles: the red rectangles must have blocked the greens. But the only way

to block the green rectangles from reaching the vertical edge is by a solid barrier of red rectangles. But a solid barrier of red rectangles must join the two horizontal edges. So either the greens span the vertical edges or the reds span the horizontal edges. (Anyone who is familiar with the game Hex may see comparisons here.)

(Incidentally, while the essence of the above paragraph is a very intuitive statement, actually proving it formally and topologically requires some work. Briefly: the set of all green areas can be divided into connected subsets. Assuming that none of these subsets straddle both vertical edges, consider the union of the left vertical edge together with all green connected subsets that touch on the left vertical edge. Then a small strip touching the boundary of this set on the outside will be coloured red, and this red strip defines a red set of rectangles which will straddle the two horizontal edges.)

Now there is a small matter of checking that chains of green rectangles spanning the vertical edges does ensure that the big rectangle has a horizontal integer edge. The only real problems are superfluous rectangles in a chain, which are easily ditched; rectangles that touch only on a corner, which is also not a problem; and backwards-going chains, which are easily dealt with too (we are subtracting integers instead of adding them on, but the grand total will always be an integer).