Tiling rectangles

Dedicated to Jan von Plato, for his 50th birthday

Introduction

We present a general result about tiling rectangles, one corollary being the theorem that, whenever a rectangle is tiled by rectangles each of which has at least one integral side, then the tiled rectangle has at least one integral side. One clever proof of this last theorem is to use integration: consider the function

$$(x, y) \longmapsto e^{2i\pi x}e^{2i\pi y}$$

the integral of this function over a rectangle is 0 iff at least one side of the rectangle has one integral side. Furthermore, the integral over a rectangle is the sum of the integral over the tiling subrectangles. Hence if all the integral over the tiling subrectangles are 0 so is the integral over the total rectangle. This argument, besides being clever, opens some interesting question of the logical status of integers. Should it be a primitive notions, or can it be that the concept of integers is best approched from what appears first as more sophisticated mathematical notions? For instance, one may think as defining integers as periods of the function

$$x \longmapsto e^{2i\pi x}$$

(Other definitions of integers have been considered, one being Fredholm index of compact operators.) This motivated me to look for a purely elementary proof of the theorem on tiling rectangles, which may be considered in some way as a "constructive version" of the argument using integration. I did spend a long time looking for a purely "logical" proof, trying to use some tensor products of Boolean algebra or lattices, until I realised, thanks to Henri Lombardi, that what I was looking for is not captured by pure logic, but requires some algebra, here some elementary linear algebras and simple properties of tensor products of Z-modules.

There is a nice survey of different possible proofs for the theorem on tiling rectangles in [Wag] and the argument that follows may be one sixteenth proof for this theorem [Key].

1 Decomposition of polynomials

We suppose that $a_1, \dots, a_n$ and $b_1, \dots, b_m$ are indeterminates. We write $a_I = \Sigma_{i \in I} a_i$ and $b_J = \Sigma_{j \in J} b_j$ for $I \subseteq [1, n]$ and $J \subseteq [1, m]$. We write $A = a_{[1,n]}$ and $B = b_{[1,m]}$.

Let P be the free Z-module on $a_1, \dots, a_n$, and let Q be the free Z-module on $b_1, \dots, b_m$.

Lemma: Let $e_1, \dots, e_n$ be another Z-basis of P and $f_1, \dots, f_m$ another Z-basis of Q. If

$$\Sigma r_{ij} e_i f_j = \Sigma s_{ij} e_i f_j$$

then $r_{ij} = s_{ij}$.

Proof: We can write $e_i = \Sigma u_{ik} a_k$ and $f_j = \Sigma v_{jl} b_l$ where the matrices (u_{ik}) and (v_{jl}) are invertible. We have then

$$\Sigma(\Sigma u_{ik} v_{jl} r_{ij}) a_k b_l = \Sigma(\Sigma u_{ik} v_{jl} s_{ij}) a_k b_l$$

and hence, since $a_k,\ b_l$ are indeterminates

$$\Sigma u_{ik} v_{jl} r_{ij} = \Sigma u_{ik} v_{jl} s_{ij}$$

Since the matrices (u_{ik}) and (v_{jl}) are invertible this implies $r_{ij} = s_{ij}$.

What we are using implicitely is the notion of tensor product of two Z-modules. But we have a concrete representation of this product by using indeterminates (like in 19th century algebra).

Theorem: Suppose $AB = \Sigma_{\alpha \in W} a_{I_\alpha} b_{J_\alpha}$ with I_α of the form $[p, q] \subseteq [1, n]$ and J_α of the form $[r, s] \subseteq [1, m]$. For any partition U, V of W we have that

- A is a linear combination of $a_{I_\alpha},\ \alpha \in U$ with integer coefficients or
- B is a linear combination of $b_{J_\alpha},\ \alpha \in V$ with integer coefficients.

Proof: Let P_1 be the submodule generated by $a_{I_\alpha},\ \alpha \in U$. Similarly, let P_1 be the submodule generated by $b_{J_\alpha},\ \alpha \in V$.

The theorem says that $A \in P_1$ or $B \in Q_1$.

I claim that we can find another basis $e_1, \ldots, e_n$ of P and $k \leq n$ such that $P_1 = Ze_1 + \ldots + Ze_k$.

Indeed the submodule generated by $a_{I_\alpha},\ \alpha \in U$ is generated by a subfamily $a_{I_1}, \ldots, a_{I_k}$ with $I_1 = [p_1, q_1], \ldots, I_k = [p_k, q_k]$ such that $p_1 < \ldots < p_k$ (if both $a_{[p,q_1]}$ and $a_{[p,q_2]}$ appear, with $q_1 < q_2$ we can replace $a_{[p,q_2]}$ by $a_{[q_1+1,q_2]}$). We can then take $e_1 = a_{I_1}, \ldots, e_k = a_{I_k}$ and complete this to a basis of P.

Similarly we can find another basis $f_1, \ldots, f_m$ of Q and $l \leq m$ such that $Q_1 = Zf_1 + \ldots + Zf_l$.

We write $A = \Sigma x_i e_i$ and $B = \Sigma y_j f_j$. By hypothesis we can find $d_i \in Q,\ c_j \in P$ such that

$$AB = \Sigma x_i y_j e_i f_j = \Sigma_{i \leq k} e_i d_i + \Sigma_{j \leq l} c_j f_j \in P_1 Q + P Q_1$$

and hence, by the lemma, $x_i y_j = 0$ if $i > k$ and $j > l$. Hence we have $x_i = 0$ for all $i > k$ or $y_j = 0$ for all $j > l$. Hence $A \in P_1$ or $B \in Q_1$.

2 Decomposition of rectangles

As direct applications, we have the following two results.

Theorem: If a rectangle R of side A, B is tiled by rectangle $R_\alpha,\ \alpha \in W$ of side a_α, b_α then, for any partition U, V of W we have that

- A is a linear combination of $a_\alpha,\ \alpha \in U$ with integer coefficients or
- B is a linear combination of $b_\alpha,\ \alpha \in V$ with integer coefficients.

Corollary: Whenever a rectangle is tiled by rectangles each of which has at least one integral side, then the tiled rectangle has at least one integral side.

References

[Key] R. Kenyon. A note on tiling with integer-sided rectangles. J. Combin. Theory Ser. A 74 (1996), no. 2, 321–332.

[Wag] S. Wagon. Fourteen proofs of a result about tiling a rectangle. Amer. Math. Monthly 94 (1987), no. 7, 601–617.