Q, and then reflecting Q through C to S; this last point S would be brought by the rotation $2 \wedge$ CD to the position T, namely the reflexion of Q with respect to D; and so on, till after arriving at the reflexion W of Q, relatively to the last corner G of the given polygon, we should be brought *back* from W to the original position P, by the final rotation $2 \wedge$ GA; because P is the reflexion of Q, with respect to the first given corner A. (Arcs of *small* circles are denoted in the present figure by straight and *dotted lines;* arcs of *great* circles by lines *without* dots, but still, for simplicity, *straight.*)

346. Again consider the equation of art. 280,

$$\gamma^z\beta^y\alpha^x = -1,$$

which gives,

$$\beta^y\alpha^x = -\gamma^{-z},$$

and, therefore, by the associative principle, and by the property (192) of the reciprocal of a product,

$$\beta^y \, . \, \alpha^x\rho\alpha^{-x} \, . \, \beta^{-y} = \gamma^{-z}\rho\gamma^z.$$

In interpreting this equation, in connexion with fig. 56, of art. 280, on the plan of art. 341, we are led to introduce, what it is extremely easy to form, the conception of SPHERICAL ANGLES as REPRESENTING CONICAL ROTATIONS. In fact, if ABC be any spherical angle, it is natural, when once we combine the conception of such an angle, with the conception of a conical rotation, to regard the latter as being the operator which would change, by a *plane* rotation, the *tangent* to the side BA of the given angle ABC, to the tangent to the *other* side BC of the same spherical angle. Now the last written formula of the present article is easily seen to express, that if the rotation round the pole A (in the lately cited fig. 56), through the angle $x\pi$, be followed by a rotation round the pole B (in the same figure) through an angle $= y\pi$, the result will be equivalent to a rotation round the pole C, through an angle $= -z\pi$. But the angles of the triangle ABC (in the same figure) were:

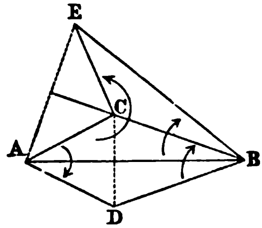
$$A = \tfrac{1}{2}x\pi; \quad B = \tfrac{1}{2}y\pi; \quad C = \tfrac{1}{2}z\pi.$$

If then, for any spherical triangle, ABC, the double of the rotation represented by the angle CAB be followed by the double of the rotation represented by the angle ABC, the result will be the double of the rotation represented by the angle ACB (which latter is the opposite of the rotation BCA).

347. To shew this geometrically, let D and E be chosen so (see the annexed figure 77) that we may have the following equations between angles,

DBA = ABC = CBE, CAB = BAD, ACB = BCE;

Fig. 77.



and let us take as *two operand points*, to be separately and successively employed, the vertex C, and the base corner A, of the spherical triangle ABC. Operating then first on the vertex C, by the two successive rotations,

$$2 \times C\hat{A}B, \text{ and } 2 \times A\hat{B}C,$$

or by

$$C\hat{A}D \text{ and } D\hat{B}C,$$

we change C first to D, and then back to C again; but such would have also been the final result, so far as the operand point C is concerned, of *any* rotation whatever round that point C itself as a pole; and, therefore, in particular, such would have been the result, relatively to *this* operand C, of the rotation represented by

$$2 \times A\hat{C}B.$$

Again, as a new and independent process, let us begin with the base-corner A as an operand point. The first component rotation,

$$2 \times C\hat{A}B,$$

being performed round this point A as a pole, leaves *its* position undisturbed. The second component and conical rotation, represented by

$$2 \times A\hat{B}C,$$

transfers the new operand point A to E. But it is clear, from the figure, that the same transference might also be effected, by a rotation round the vertex C as a pole, represented by

$$2 \times \text{A}\hat{\text{C}}\text{B}.$$

The theorem of the last article is therefore seen to be true, for the TWO *different operand points*, C and A: whence it is easily seen, by the general *conception of rotation*, to be valid for *all others* also. (An inspection of figs. 52, 57, of articles 269, 281, may serve slightly to illustrate this result.)

348. An important although particular case, of the general theorem of rotation contained in the two last articles, is illustrated by fig. 43, of art. 242: namely, the case where the triangle ABC is *triquadrantal*. In such a case, because a conical rotation through a doubled right angle is equivalent to a *reflexion* with respect to the axis or pole, we may expect to find from the general theorem, that "TWO *successive reflexions, relatively to* TWO *rectangular axes, are equivalent to a* SINGLE *reflexion, with respect to a* THIRD *axis perpendicular to both the former*." And accordingly we see in fig. 43, that if E be first reflected with respect to A to F, and if F be then reflected with respect to B to D, the final result is the same as if E had been at once reflected with respect to C (to D). It is clear also that, in this *case*, of TRI-RECTANGULARITY, *three successive reflexions* (with respect to any three rectangular axes), produce, *on the whole*, NO CHANGE: a conclusion which answers geometrically to the formulæ (210),

$$ijk = -1, \quad kji = +1;$$

because these give, for *any operand vector* ρ, the identities,

$$ijk\rho k^{-1}j^{-1}i^{-1} = kji\rho i^{-1}j^{-1}k^{-1} = \rho.$$

349. More generally, from the results of the two foregoing articles, or from the lately cited formula of art. 280, namely

$$\gamma^z\beta^y\alpha^x = -1,$$

which gives the equation,

$$\gamma^z\beta^y\alpha^x\rho\alpha^{-x}\beta^{-y}\gamma^{-z} = \rho,$$

we may infer, on the same general plan of interpretation (341), that *three successive rotations*, represented respectively by the DOUBLES *of three successive angles of any spherical triangle*, for instance (see fig. 56), by

$$2\text{C}\hat{\text{A}}\text{B}, \quad 2\text{A}\hat{\text{B}}\text{C}, \quad 2\text{B}\hat{\text{C}}\text{A},$$

produce, on the whole, NO EFFECT. And it is easy to generalize still farther this result, so as to prove the following theorem: "If a *body* B be made to *revolve* through *any number of successive and finite rotations*, represented as to their axes and amplitudes by the DOUBLES OF THE ANGLES, $A_1, A_2, \ldots A_n$, *of any spherical polygon*, this body B will be BROUGHT BACK, hereby, *to its own original position*." You will find, by the printed Proceedings of the Royal Irish Academy, that I stated this Theorem (with only a slight difference in its wording), at a general meeting of that Academy, in November, 1844, as a consequence of those principles respecting Quaternions, which had been communicated to the Academy by me, about a year before. The theorem, at that time, appeared to me to be new; nor am I able, at this moment, to specify any work in which it may have been anticipated: although it seems to me *likely* enough that some such anticipation may exist. Be that as it may, the theorem was certainly *suggested to me* by the quaternions; nor can I easily believe that *any other* mathematical method shall be found to furnish any SIMPLER form of EXPRESSION for the same general geometrical result. For there is little difficulty in seeing that the theorem coincides substantially with the conclusion of art. 345; and may, therefore, be *expressed* in this calculus by the same IDENTITY,

$$\frac{\alpha}{\kappa}\frac{\kappa}{\iota}\cdots\frac{\delta}{\gamma}\frac{\gamma}{\beta}\frac{\beta}{\alpha}=1.$$

350. But it is worth while to inquire what will happen, if *instead* of compounding, as in some recent articles, rotations represented by the DOUBLES of the sides of a spherical triangle, or polygon, we compound rotations represented by the SIDES THEMSELVES of the figure; and with respect to *this* inquiry, the Calculus of Quaternions has conducted to results which, although not very difficult otherwise to *prove*, appear to me less *likely* to have been anticipated.

It has been shewn, in the present Lecture (arts. 258 to 263), that the product

$$q=(\delta\epsilon^{-1})^{\frac{1}{2}}\ (\epsilon\zeta^{-1})^{\frac{1}{2}}\ (\zeta\delta^{-1})^{\frac{1}{2}},$$

of the square roots of the successive quotients,