THE

LONDON, EDINBURGH, AND DUBLIN

PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE.

CONDUCTED BY

SIR DAVID BREWSTER, K.H. LL.D. F.R.S.L.&E. &c.
RICHARD TAYLOR, F.L.S. G.S. Astr.S. Nat.H.Mosc. &c.
RICHARD PHILLIPS, F.R.S.L.&E. F.G.S. &c.
SIR ROBERT KANE, M.D. M.R.I.A.
WILLIAM FRANCIS, PH.D. F.L.S. F.C.S.

"Nec aranearum sane textus ideo melior quia ex se fila gignunt, nec noster vilior quia ex alienis libamus ut apes." JUST. LIPS. *Polit.* lib. i. cap. 1. Not.

VOL. I.—FOURTH SERIES.

JANUARY—JUNE, 1851.

LONDON.

RICHARD TAYLOR, RED LION COURT, FLEET STREET,
Printer and Publisher to the University of London;
SOLD BY LONGMAN, BROWN, GREEN, AND LONGMANS; SIMPKIN, MARSHALL AND CO.; S. HIGHLEY; WHITTAKER AND CO.; AND SHERWOOD, GILBERT, AND PIPER, LONDON: — BY ADAM AND CHARLES BLACK, AND THOMAS CLARK, EDINBURGH; SMITH AND SON, GLASGOW; HODGES AND SMITH, DUBLIN; AND WILEY AND PUTNAM, NEW YORK.

"Meditationis est perscrutari occulta; contemplationis est admirari perspicua . . . . . . Admiratio generat quæstionem, quæstio investigationem, investigatio inventionem."—*Hugo de S. Victore.*

—"Cur spirent venti, cur terra dehiscat,
Cur mare turgescat, pelago cur tantus amaror,
Cur caput obscura Phœbus ferrugine condat,
Quid toties diros cogat flagrare cometas;
Quid pariat nubes, veniant cur fulmina cœlo,
Quo micet igne Iris, superos quis conciat orbes
Tam vario motu."

J. B. Pinelli ad Mazonium.

PREFACE TO THE FOURTH SERIES.

In commencing with the year 1851 a New Series of The Philosophical Magazine, the Editors are induced to hope that they may refer to the progress and conduct of the work through its existence of more than half a century. Having been begun by Dr. Alexander Tilloch in 1798, with a view to the promotion of science, it has ever been the object both of its original Editor and of his successors, to record all the great discoveries made, since the æra of its commencement, as they could be collected from the most authentic sources foreign as well as domestic.

In addition to the copious supply which these afford, the numerous Original Communications with which the Philosophical Magazine has been honoured by the most eminent Cultivators of Science of the present day may be stated as giving the work a just claim to general support. As the representative also of the several Journals* which have from time to time merged in this, it has long occupied in all parts of the world a rank similar to that held by the *Annales de Chimie et de Physique*, by Poggendorff's *Annalen*, and others; whilst from its extensive foreign circulation it is the principal vehicle of communication between the Philosophers of our own and of foreign countries, as is shown by the constant reference to its pages, and the frequent translation of its Articles, in foreign scientific works.

The important duty of making known in this country

* Nicholson's Journal: The Annals of Philosophy: The Journal of Science.

the labours and discoveries in the various branches of Natural Philosophy abroad, has, it is hoped, been to a considerable extent fulfilled in the great number of Translations and Abstracts which have been given from the principal foreign Journals and Transactions, with a view to enable the reader to keep pace with the progress of science in every stage of its advancement. In this, as well as in other departments of the Journal, the Editors will now have the constant aid of Dr. William Francis, the Editor of the 'Chemical Gazette,' to whose services they have long been greatly indebted, and with whose assistance arrangements are made for introducing material improvements in the New Series, and particularly a more regular and fuller account of scientific discoveries in Foreign Countries.

The commencement of this New Series suggests the hope that the Philosophical Magazine may at this period receive an accession to the number of its supporters. Those of its Editors who have stood beside it for half a century, and made it their endeavour that it should be honestly, independently and usefully conducted, may be permitted, on this occasion, to urge how much the means of giving additional interest and value to the Journal must depend upon the support afforded to them; in the hope that many lovers of science who are not already subscribers may take this opportunity of adding to the number of those by whose encouragement alone the work has been upheld.

In acknowledging the favours of their Correspondents, the Editors confidently request a continuance of them as the best means of insuring their future success.

Richard Taylor.

Jan. 1, 1851.

CONTENTS OF VOL. I.

(FOURTH SERIES.)

NUMBER I.—JANUARY 1851.

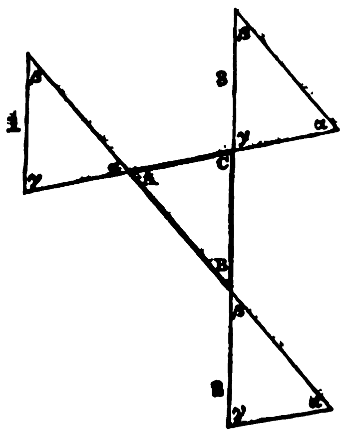
NUMBER II.—FEBRUARY.

NUMBER III.—MARCH.

XXV. *On the Geometrical Theory of Rotation.* By W. F. Donkin, *M.A. &c., Savilian Professor of Astronomy in the University of Oxford**.

Although two demonstrations† have already appeared in this Journal of the *triangle of rotations* (to adopt Mr. Sylvester's convenient designation), I think it worth while to add the following, which, though not substantially different, exhibits the theorem in the simplest way, and under the most striking aspect. I shall also subjoin some further illustrations of its use in connexion with quaternions.

Fig. 1.



Let ABC, fig. 1, be any triangle on a sphere fixed in space, and $\alpha\beta\gamma$ a triangle on an equal and concentric sphere, moveable about its centre. The sides and angles of $\alpha\beta\gamma$ are equal to those of ABC, but differently arranged, one triangle being the inverse or reflexion of the other. [In the figure straight lines are used for convenience to represent arcs of great circles.]

Now if the triangle $\alpha\beta\gamma$ be placed in the position 1, so that the sides containing the angle α may be in the same great circles with those containing A, it is obvious that it may slide along AB into the position 2, and then along BC into the position 3; into which last position it might also be brought by sliding along AC. Hence, denoting the rotations by arcs on the *fixed* sphere, we have the following theorem:—

I. *Twice the rotation* AB *followed by twice the rotation* BC *produces the same displacement as twice the rotation* AC.

Or, denoting them by arcs on the moving sphere,—

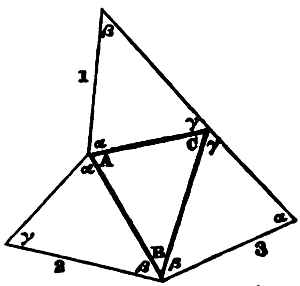
Twice the rotation $\beta\alpha$ followed by twice the rotation $\gamma\beta$ produces the same displacement as twice the rotation $\gamma\alpha$.

Again, in fig. 2, if we denote the centre of the sphere by O, it is obvious that the pyramid whose intersection with the surface of the sphere is the triangle $\alpha\beta\gamma$, might pass from position 1 into position 2, by turning round OA through an angle $2\pi-2A$,

* Communicated by the Author.

† See Phil. Mag. for June and December 1850, and January 1851.

Fig. 2.



and then into position 3 by turning round OB through an angle $2\pi-2B$; into which position it might also have been brought from 1 by turning round OC (in the contrary direction) through an angle $2\pi-2C$. Hence we have what may be called, with reference to theorem I., the *polar* theorem; namely, as regards the fixed sphere,—

II. *A positive rotation* $2(\pi-A)$ *round* OA, *followed by a positive rotation* $2(\pi-B)$ *round* OB, *produces the same displacement as a negative rotation* $2(\pi-C)$ *round* OC.

The enunciation of the theorem is the same as regards the *moving* sphere, if $\alpha\beta\gamma$ be put for ABC. But it must be carefully observed, that on the fixed sphere the arrangement of ABC is such that C is the *positive* pole of a rotation from CA to CB; whilst, on the moving sphere, γ is the *negative* pole of a rotation from $\gamma\alpha$ to $\gamma\beta$.

It is obvious that a perfectly similar demonstration may be employed in the case of a polygon. As regards fig. 2, this has been given by Mr. Sylvester. Also, if we take the moving polygon equilateral but *not equiangular* with the fixed polygon, and then diminish the sides indefinitely, we arrive, in the case of fig. 2, at Poinsot's mode of representing the most general kind of rotatory motion; whilst in the case of fig. 1, taking the polygons equiangular but not equilateral, we obtain two cones, the reciprocals of Poinsot's cones, one of which *slides* (without rolling) upon the other. Thus the two theorems given above are, in fact, particular cases of the general relations between the curvatures of these cones and the angular velocities of the body about the instantaneous axis, and of the instantaneous axis in the body or in space.

In the Philosophical Magazine for June 1850, I showed that a comparison of the preceding theorems with the results of a particular mode of interpreting quaternions (explained in the July Number) enabled us to account, *à priori*, for the connexion observed by Mr. Cayley between certain quaternion formulæ and those which occur in the theory of the rotation of solids. (Phil. Mag. Ser. 3. vol. xxxiii. p. 196.)

In fact, since the quaternion $\cos\theta+\sin\theta(il+jm+kn)$ may be considered to represent the rotation of a radius vector through an angle θ round an axis whose direction cosines are l, m, n; or, which comes to the same thing, the description of a corresponding arc on the surface of a sphere; and since the descriptions of arcs on the sphere correspond to and define the effect of rotations of a solid round the same axes, but through double the angles, it follows from these theorems that the quaternion

$$\cos\frac{\theta}{2}+\sin\frac{\theta}{2}(il+jm+kn) \quad . \quad . \quad . \quad . \quad (1.)$$

represents the positive rotation of a solid through an angle θ, round an axis whose direction cosines are l, m, n referred to axes *fixed in space*; whilst

$$\cos\frac{\theta}{2}-\sin\frac{\theta}{2}(il+jm+kn) \quad . \quad . \quad . \quad . \quad (2.)$$

may be interpreted similarly with reference to axes *fixed in the body*.

Mr. Boole, in noticing the interpretation of the former of these expressions (Phil. Mag. vol. xxxiii. p. 279), observes that a quaternion $w+ix+jy+kz$, whose constituents do not satisfy the condition $w^2+x^2+y^2+z^2=1$, is not directly interpretable in geometry. This, however, is certainly not true with reference to interpretation by means of the rotation of *lines*; for in that case the quaternion aq (where q does satisfy the above condition, and a is a numerical coefficient) represents (as I have shown) the rotation of a radius vector, combined with an *alteration of its length* in the ratio of a to 1. And I think it not impossible that some corresponding interpretation of the coefficient may be discovered in the case of the solid.

At present, however, I proceed to illustrate the subject by one or two additional examples.

In all that follows, $\xi\eta\zeta$ will refer to axes fixed in space, and xyz to axes fixed in the rotating body.

Let p, q, r have their usual significations in the theory of rotation, and $w^2=p^2+q^2+r^2$. Also let the position of the body at any instant be defined by the values of λ, μ, ν; where

$$\lambda=l\tan\frac{\theta}{2}, \quad \mu=m\tan\frac{\theta}{2}, \quad \nu=n\tan\frac{\theta}{2},$$

and θ is the angle through which the body would have to turn round an axis whose direction cosines are l, m, n, in order to pass from the position in which the two sets of axes coincide, into that in which it actually is at the instant considered. To adopt the names proposed by Mr. Cayley and Mr. Sylvester, $\lambda\mu\nu$ are the *coordinates of the resultant rotation*, and l, m, n are the direction cosines of the *axis of displacement*. Lastly, let

$$\sec^2\frac{\theta}{2}=1+\lambda^2+\mu^2+\nu^2=\kappa.$$

Let it be required to express p, q, r in terms of $\lambda\mu\nu$ and their differential coefficients.

It is to be observed that l, m, n refer indifferently to either set of axes. Considering them to refer to x, y, z, we have the expression (2.), namely,

$$\cos\frac{\theta}{2}-\sin\frac{\theta}{2}(il+jm+kn),$$

for the quaternion representative of the displacement. In the next instant dt, a rotation wdt takes place round an axis whose direction cosines are $\frac{p}{w}, \frac{q}{w}, \frac{r}{w}$; which will be represented by the quaternion

$$\cos\frac{wdt}{2}-\sin\frac{wdt}{2}\cdot\left(i\frac{p}{w}+j\frac{q}{w}+k\frac{r}{w}\right);$$

or, neglecting quantities of the second order, by

$$1-\frac{dt}{2}(ip+jq+kr).$$

And the displacement of the body in its new position may be represented either by the product

$$\left(1-\frac{dt}{2}(ip+jq+kr)\right)\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}(il+jm+kn)\right),$$

or by the quaternion

$$\cos\frac{\theta'}{2}-\sin\frac{\theta'}{2}(il'+jm'+kn'),$$

where $\theta', l', \ldots$ are put for $\theta+d\theta, l+dl, \ldots$. Equating these two expressions, and then applying each side of the equation as a multiplier to

$$\cos\frac{\theta}{2}+\sin\frac{\theta}{2}(il+jm+kn),$$

we get

$$1-\frac{dt}{2}(ip+jq+kr)=\cos\frac{\theta'}{2}\cos\frac{\theta}{2}(1-i\lambda'-j\mu'-k\nu')(1+i\lambda+j\mu+k\nu),$$

of which the right side, developed (omitting terms of the second order, and observing that $2(\lambda d\lambda+\mu d\mu+\nu d\nu)=d\kappa$), is easily found to be

$$1-\frac{1}{\kappa}\{i(d\lambda+\nu d\mu-\mu d\nu)+j(d\mu+\lambda d\nu-\nu d\lambda)+k(d\nu+\mu d\lambda-\lambda d\mu)\},$$

which, compared with the expression on the left side, gives

$$\kappa p = 2\left(\frac{d\lambda}{dt} + \nu\frac{d\mu}{dt} - \mu\frac{d\nu}{dt}\right),$$

with symmetrical expressions for q and r.

For a deduction of these expressions by direct differentiation, see Mr. Cayley's paper in the Cambridge Mathematical Journal, vol. iii. p. 227. It is to be observed that the method employed above does not require a knowledge of the expressions for λ, μ, ν, or p, q, r, in terms of the nine direction cosines of one set of axes referred to the other.

Lastly, let it be required to find the relations between λ, μ, ν, and the ϑ, ϕ, ψ of the ordinary theory. For convenience of description, let the earth be the body considered, the axes of x and y being fixed in the plane of the equator, and that of z coinciding with the polar axis, whilst the axes of ξ and η are fixed in the plane of the ecliptic. Then ϑ is the inclination of the equator to the ecliptic; ϕ is the right ascension of the axis of x; and ψ is the longitude of the ascending node of the equator, reckoned from the axis of ξ.

Now it is obvious that the earth might have been brought from the position in which the two sets of axes coincide, into that in which it actually is at any instant, by three successive rotations; namely, first, a rotation ψ round the axis of z; second, a rotation ϑ round the axis of x; and third, a rotation ϕ round the axis of z again. Hence the principles above established give the following equation:

$$\kappa^{-\frac{1}{2}}(1-i\lambda-j\mu-k\nu)=\left(\cos\frac{\phi}{2}-k\sin\frac{\phi}{2}\right)\left(\cos\frac{\vartheta}{2}-i\sin\frac{\vartheta}{2}\right)\left(\cos\frac{\psi}{2}-k\sin\frac{\psi}{2}\right),$$

of which the second side developed and compared with the first, gives

$$\kappa=\sec^2\frac{\vartheta}{2}\cdot\sec^2\frac{\psi+\phi}{2},$$

$$\lambda=\tan\frac{\vartheta}{2}\cdot\frac{\cos\frac{\psi-\phi}{2}}{\cos\frac{\psi+\phi}{2}},\qquad \mu=\tan\frac{\vartheta}{2}\cdot\frac{\sin\frac{\psi-\phi}{2}}{\cos\frac{\psi+\phi}{2}},$$

$$\nu=\tan\frac{\psi+\phi}{2}.$$

The inverse expressions are easily obtained; thus we have

$$\tan\frac{\psi-\phi}{2}=\frac{\mu}{\lambda},\quad \tan\frac{\psi+\phi}{2}=\nu,$$

whence

$$\tan\psi=\frac{\mu+\nu\lambda}{\lambda-\mu\nu},\quad \tan\phi=\frac{\nu\lambda-\mu}{\lambda+\mu\nu},$$

and

$$\sec^2\frac{\vartheta}{2}=\frac{\kappa}{1+\nu^2},$$

which may be verified by means of the values of ϑ, ϕ, ψ, λ, μ, ν in terms of the nine direction cosines.

Oxford, Jan. 11, 1851.

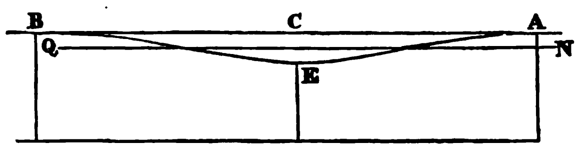
XXVI. *On the Negative Wave of Translation.* *By* A. J. Robertson, *Civil Engineer**.

In a paper on the Positive Wave of Translation, in the December Number of this Magazine, it was shown that the distance the crest of the wave moves through, during the time that an elementary column makes a vertical oscillation from rest to rest, is $L+\frac{V}{a}$,—the quantity $\frac{V}{a}$ being half the absolute translation of the particles in space,—and that in consequence the velocity of the wave is increased from $\sqrt{(a+k)g}$ to

$$\frac{L+\frac{V}{a}}{L}\sqrt{(a+k)g}=\sqrt{\frac{a+k}{a}(a+2k)g}.$$

In the negative wave of translation, which is generated by the *abstraction* of a quantity of water from the Channel, and which is a hollow instead of a swell, the translation of the particles is in the direction *contrary* to the motion of the wave. Hence the crest of the wave moves, during the vertical oscillation of a column, through a space *less* than L by a quantity $\frac{V}{a}$.

Applying the same principles to the negative as to the positive



wave, retaining the same symbols, and remembering that k is negative, we obtain

* Communicated by the Author.